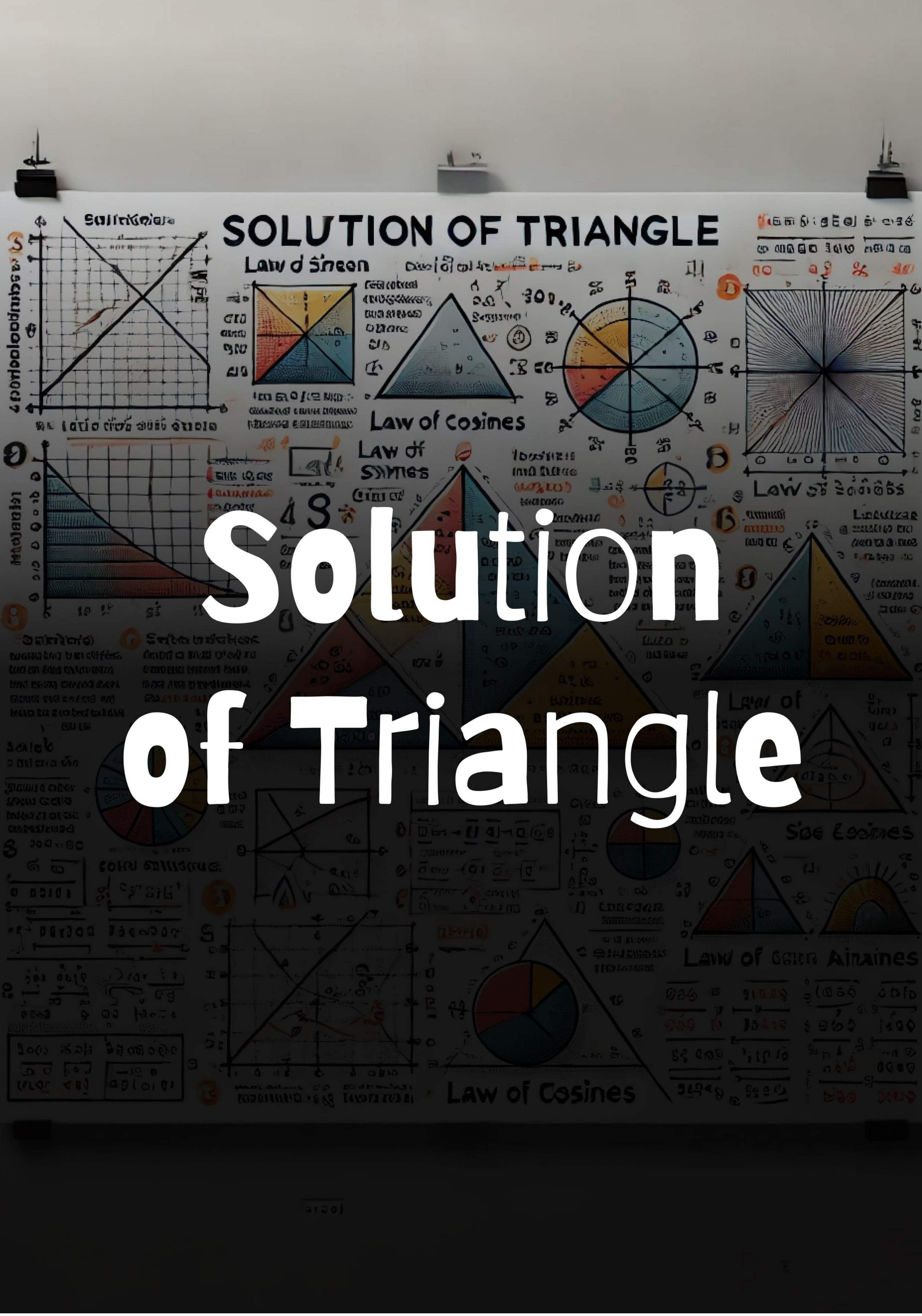
Solution of Triangle



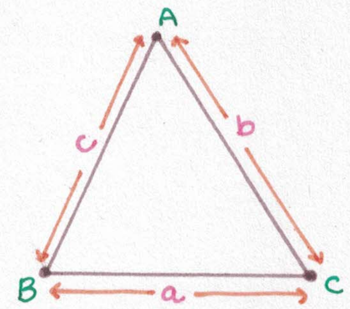
SOLUTION OF TRIANGLE

Elements of a Triangle:

$\angle A, \angle B, \angle C,$

a, b, c

$0^\circ < \angle A, \angle B, \angle C < 180^\circ$



1 $a+b>c$, $b+c>a$, $c+a>b$

The sum of 2 sides of any Δ is greater than third side.

2 $a>|b-c|$, $b>|a-c|$, $c>|a-b|$

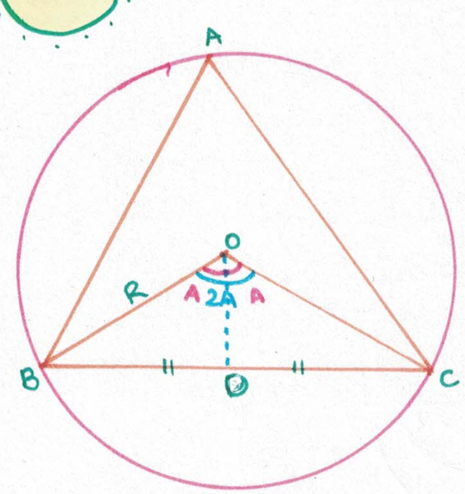
The difference of two sides of any Δ is less than the third sides.

Relation b/w sides and Triangles

1 Sine Rule

$$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \text{ (circum radius)}}$$

PROOF



In Δ OBD,

$\sin A = \frac{BD}{OB}$

$\sin A = \frac{BD}{R}$

$BD = R \sin A$

$BC = 2BD$

$BC = 2R \sin A$

$a = 2R \sin A$

$a/\sin A = 2R$

Similarly,

$$\frac{b}{\sin B} = 2R$$

$$\frac{c}{\sin C} = 2R$$

$$\Rightarrow \boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R}$$

Ques: In ΔABC, $a=3$ $\angle A = 30^\circ$, $\angle B = 60^\circ$. Find b and c?

Sol: $\frac{a}{\sin A} = \frac{3}{1/2} = 6$

$6 = \frac{b}{\sin B} \Rightarrow b = 6 \times \frac{\sqrt{3}}{2} = 3\sqrt{3}$

$$\frac{c}{\sin C} = \frac{a}{\sin A} = \frac{b}{\sin B}$$

$c = a$ $\quad$ $b = c$

$c = 6$

Ques: Find the ratio of sides when $\angle A = 30^\circ$, $\angle B = 45^\circ$

Sol: $\angle C = 180 - (30+45) = 105^\circ$

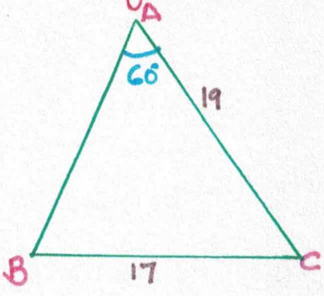
$$\frac{a}{1/2} = \frac{b}{1/\sqrt{2}} = \frac{c}{\frac{\sqrt{3}+1}{2\sqrt{2}}}$$

$$2a = \sqrt{2}b = \frac{2\sqrt{2}c}{\sqrt{3}+1}$$

$$a : b : c = \sqrt{2} : 1 : \frac{2}{\sqrt{3}+1}$$

$$a:b:c = \sqrt{3}+1 : 2 : \sqrt{2}$$

Ques: In ΔABC, $AC = 19$ cm., $BC = 17$ cm., angle $BAC = 60^\circ$. Find $\angle ABC$.



Sol: $a = 17$ $\quad$ $b = 19$ $\quad$ $A = 60^\circ$

$$\frac{a}{\sin A} = \frac{17 \cdot 2}{\sqrt{3}} = \frac{34}{\sqrt{3}} = \frac{b}{\sin B}$$

$$\sin B = \frac{19\sqrt{3}}{34}$$

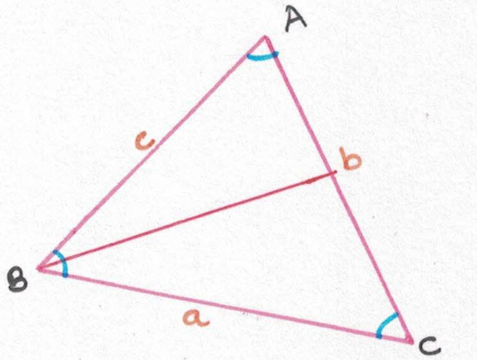
$$\Rightarrow \quad B = \sin^{-1}\left(\frac{19\sqrt{3}}{34}\right)$$

2 COSINE FORMULA

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$\cos B = \frac{c^2+a^2-b^2}{2ac}$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$



PROOF

$$AD = c\cos A$$

$$BD = c\sin A$$

$$CD = AC - AD$$

$$CD = b - c\cos A$$

In ΔBCD,

$$BD^2 + CD^2 = BC^2$$

$$(c\sin A)^2 + (b - c\cos A)^2 = a^2$$

$$c^2\sin^2 A + b^2 + c^2\cos^2 A - 2bc\cos A = a^2$$

$$b^2 + c^2 - 2bc\cos A = a^2$$

$$\boxed{\cos A = \frac{b^2+c^2-a^2}{2bc}}$$

3 COT FORMULA

Now,

$$a^2 + c^2 = 2b^2$$

$$\cot A = \frac{\cos A}{\sin A} = \frac{b^2+c^2-a^2}{2bc \quad 2Ra} = \frac{b^2+2b^2-a^2-a^2}{2bc \quad 2Ra}$$

$$\cot B = \frac{\cos B}{\sin B} = \frac{a^2+c^2-b^2}{2Rb \cdot 2ac}$$

$$\cot C = \frac{\cos C}{\sin C} = \frac{b^2+a^2-c^2}{2Rc \cdot 2ba}$$

$\cot A + \cot C$

$$= \frac{b^2+c^2-a^2}{4R\,abc} + \frac{b^2+a^2-c^2}{4R\,abc}$$

$$= \frac{2b^2}{4R\,abc}$$

$$= 2\cot B$$

Hence, $\cot A$, $\cot B$, $\cot C$ are in A.P.

Ques: Solve, $(b^2-c^2)\cot A + (c^2-a^2)\cot B + (a^2-b^2)\cot C$

Sol: $(b^2-c^2)\dfrac{\cos A}{\sin A} + (c^2-a^2)\dfrac{\cos B}{\sin B} + (a^2-b^2)\dfrac{\cos C}{\sin C}$

$$= (b^2-c^2)\left(\frac{b^2+c^2-a^2}{2bc}\right)\cdot\frac{2R}{a} + (c^2-a^2)\left(\frac{a^2+c^2-b^2}{2ac}\right)\cdot\frac{2R}{b}$$
$$+ (a^2-b^2)\left(\frac{a^2+b^2-c^2}{2ab}\right)\cdot\frac{2R}{c}$$

$$= \frac{2R}{2abc}\left[(b^2-c^2)(b^2+c^2-a^2) + (c^2-a^2)(c^2+a^2-b^2) + (a^2-b^2)(a^2+b^2-c^2)\right]$$

$$= \frac{R}{abc}\left[\begin{matrix} b^4+b^2c^2-a^2b^2-b^2c^2-c^4+a^2c^2+c^4+c^2a^2-b^2c^2-a^2c^2-a^4+ \\ a^2b^2+a^4+a^2b^2-a^2c^2-b^2a^2-b^4+b^2c^2 \end{matrix}\right]$$

$$= \frac{R}{abc}(0)$$

$$= 0$$

Ques: $a\cos\dfrac{B-C}{2} = (b+c)\sin\dfrac{A}{2}$, Prove.

Sol: $a\left[\cos\frac{B}{2}\cos\frac{C}{2}+\sin\frac{B}{2}\sin\frac{C}{2}\right]$

$$\frac{a}{b+c}=\frac{\sin\frac{A}{2}}{\cos\left(\frac{B-C}{2}\right)}$$

$$\frac{a}{b+c}=\frac{2R\sin A}{2R\sin B+2R\sin C}$$

$$=\frac{\sin A}{\sin B+\sin C}$$

$$=\frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}$$

$$=\frac{\sin A/2}{\cos\left(\frac{B-C}{2}\right)}$$

$$\boxed{a\cos\left(\frac{B-C}{2}\right)=(b+c)\sin A/2}$$

NAPIER'S ANALOGY / NAPIER'S FORMULA / TANGENT RULE

$$\tan\left(\frac{B-C}{2}\right)=\frac{b-c}{b+c}\cot\left(\frac{A}{2}\right)$$

$$\tan\left(\frac{C-A}{2}\right)=\frac{c-a}{c+a}\cot\left(\frac{B}{2}\right)$$

$$\tan\left(\frac{A-B}{2}\right)=\frac{a-b}{a+b}\cot\left(\frac{C}{2}\right)$$

PROOF

$$\frac{b}{\sin A}=\frac{c}{\sin C}$$

$$\frac{b}{c}=\frac{\sin A}{\sin C}$$

Applying Compenendo dividendo,

$$\frac{b-c}{b+c}=\frac{\sin B-\sin C}{\sin B+\sin C}$$

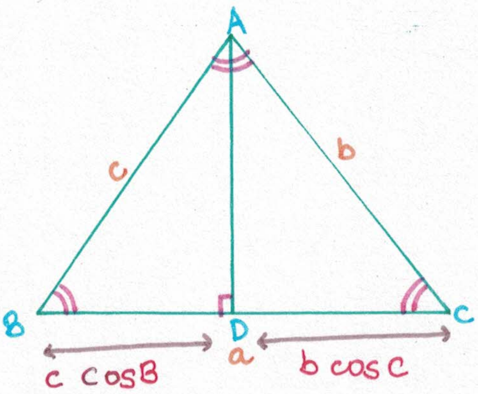
$$= \frac{2\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{B+C}{2}\right)}{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}$$

$$= \tan\left(\frac{B-C}{2}\right)\cot\left(\frac{B+C}{2}\right)$$

$$\frac{b-c}{b+c} = \tan\left(\frac{B-C}{2}\right)\tan\frac{A}{2}$$

$$\boxed{\frac{b-c}{b+c}\cdot\cot\frac{A}{2} = \tan\left(\frac{B-C}{2}\right)}$$

PROJECTION FORMULA



$$a = b\cos C + c\cos B$$

$$b = c\cos A + a\cos C$$

$$c = a\cos B + b\cos A$$

ILLUSTRATION

Solve, $b\cos^2\frac{C}{2} + c\cos^2\frac{B}{2}$ in terms of 'K', where 'K' is perimeter of ΔABC.

Here, $b\cos^2\frac{C}{2} + c\cos^2\frac{B}{2}$

$$= \frac{b}{2}[1+\cos C] + \frac{c}{2}[1+\cos B]$$

$$= \frac{b+c}{2} + \frac{1}{2}[b\cos C + c\cos B]$$

$$= \frac{b+c}{2} + \frac{1}{2}a$$

$$= \frac{a+b+c}{2} \Rightarrow b\cos^2\frac{C}{2} + c\cos^2\frac{B}{2} = K/2 \quad (K = a+b+c)$$

$= \frac{K}{2}$

HALF ANGLE FORMULA

$$\sin\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\sin\left(\frac{B}{2}\right) = \sqrt{\frac{(s-c)(s-a)}{ac}}$$

$$\sin\left(\frac{C}{2}\right) = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

Where, $s = \frac{a+b+c}{2}$, Semiperimeter

PROOF

$$1 - 2\sin^2\frac{A}{2} = \cos A$$

$$\Rightarrow 1 - \cos A = 2\sin^2\frac{A}{2}$$

$$\Rightarrow \cos A = \frac{c^2+b^2-a^2}{2bc}$$

$$\Rightarrow 1 - \frac{c^2+b^2-a^2}{2bc} = 2\sin^2\frac{A}{2}$$

$$\Rightarrow \frac{2bc - c^2 - b^2 + a^2}{2bc \times 2} = \sin^2\frac{A}{2}$$

$$\Rightarrow \frac{2bc - a^2 - (b^2 - c^2)}{4bc} = \sin^2\frac{A}{2}$$

$$\Rightarrow \frac{(a+b+c-b)(a+b+c-c)}{bc}$$

$$\Rightarrow \frac{(b+c)(a+b)}{bc}$$

$$\Rightarrow \frac{ab+b^2+ac+cb}{bc}$$

$$\Rightarrow \frac{2bc - a^2 - b^2 + c^2}{2bc} = 2\sin^2\frac{A}{2}$$

$$\Rightarrow \quad \frac{a^2 - (b-c)^2}{2bc} = 2\sin^2\frac{A}{2}$$

$$\Rightarrow \quad \sin\frac{A}{2} = \sqrt{\frac{(a+b+c)(a-b+c)}{2bc}}$$

$$\Rightarrow \quad \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\Rightarrow \quad \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$\Rightarrow \quad 2\cos^2\frac{A}{2} - 1 = \cos A$$

$$\Rightarrow \quad 2\cos^2\frac{A}{2} = 1 + \cos A$$

$$\Rightarrow \quad 2\cos^2\frac{A}{2} = 1 + \frac{b^2+c^2-a^2}{2bc} \qquad \frac{a+b+c}{2} = s$$

$$\Rightarrow \quad \cos^2\frac{A}{2} = \frac{(b+c)^2 - a^2}{4bc}$$

$$\Rightarrow \quad \cos^2\frac{A}{2} = \frac{(b+c-a)(b+c+a)}{4bc}$$

$$\Rightarrow \quad \boxed{\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}}$$

Important

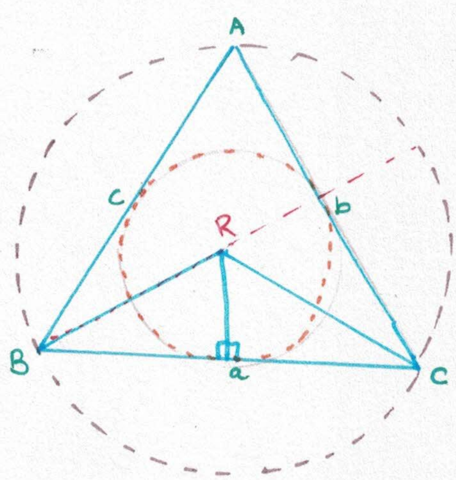
$$\Delta = \frac{1}{2}ab\sin C$$

$$4\Delta = 2ab\sin C \qquad 2\sin C = \frac{4\Delta}{ab}$$

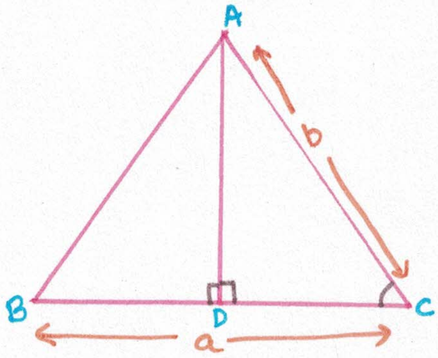
$$\frac{a}{\sin A} = \frac{c}{\sin C} = 2R$$

$$R = \frac{c}{2\sin C} = \frac{c}{\frac{4\Delta}{ab}}$$

$$\boxed{R = \frac{abc}{4\Delta}}$$



AREA OF TRIANGLE



$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{altitude}$$

$$= \frac{1}{2} \times a \times AD$$

$$= \frac{1}{2} \times a \times b \sin C$$

$$= \frac{1}{2} ab \sin C$$

$$\boxed{\text{Area}(\Delta) = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B = \frac{1}{2} ab \sin C}$$

HERON'S FORMULA

$$\boxed{\Delta = \sqrt{S(S-a)(S-b)(S-c)}}$$

Where, $S = \frac{a+b+c}{2}$ (semiperimeter)

PROOF

$$\Delta = \frac{1}{2} ab \sin C$$

$$= \frac{1}{2} ab \, 2 \sin \frac{C}{2} \cos \frac{C}{2}$$

$$= ab \sin \frac{C}{2} \cos \frac{C}{2}$$

$$= ab \sqrt{\frac{(S-a)(S-b)}{ab}} \sqrt{\frac{S(S-c)}{ab}}$$

$$= \sqrt{S(S-a)(S-b)(S-c)} \, \frac{ab}{ab}$$

$$\boxed{\Delta = \sqrt{S(S-a)(S-b)(S-c)}}$$

Area of Triangle in terms of one side & angles:

$$\Delta = \frac{1}{2} ab \sin C \qquad \begin{bmatrix} a = 2R\sin A \\ b = 2R\sin B \end{bmatrix}$$

$$\Delta = \frac{1}{2}(2R\sin A)(2R\sin B)\sin C$$

$$\Delta = \frac{(2R)^2}{2} \sin A \cdot \sin B \cdot \sin C$$

$$\Delta = \frac{1}{2}\left(\frac{a}{\sin A}\right)^2 \sin A \cdot \sin B \cdot \sin C$$

$$\boxed{\Delta = \frac{a^2}{2} \frac{\sin B \sin C}{\sin A}}$$

$$\boxed{\Delta = \frac{a^2}{2} \frac{\sin B \sin C}{\sin A} = \frac{b^2}{2} \frac{\sin A \sin C}{\sin B} = \frac{c^2}{2} \frac{\sin A \sin B}{\sin C}}$$

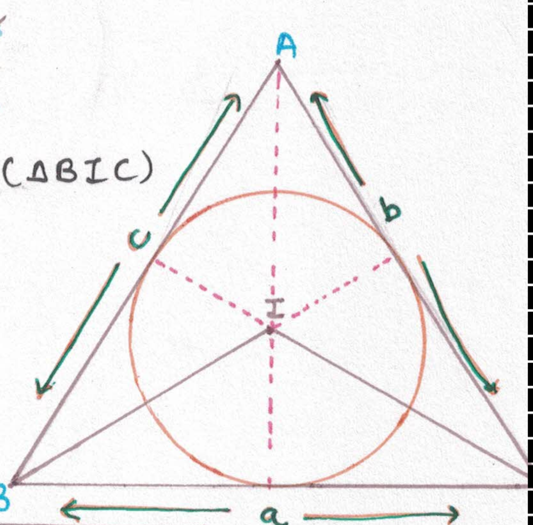
Relation between R and Δ

$$\Delta = \frac{1}{2} ab \sin C = \frac{1}{2} ab \left[\frac{c}{2R}\right] = \frac{abc}{4R}$$

$$\boxed{R = \frac{abc}{4\Delta}}$$

R is circumradius Δ → area of triangle

Relation between inradius (r) and Δ



$$\Delta = \text{area}(\Delta AIB) + \text{area}(\Delta AIC) + \text{area}(\Delta BIC)$$

$$\Delta = \frac{1}{2}(cr) + \frac{1}{2}(br) + \frac{1}{2}(ar)$$

$$2\Delta = cr + br + ar$$

$$2\Delta = (a+b+c)r$$

$$\boxed{r = \frac{2\Delta}{a+b+c}} \qquad \boxed{r = \frac{\text{area}}{\text{semiperimeter}}}$$

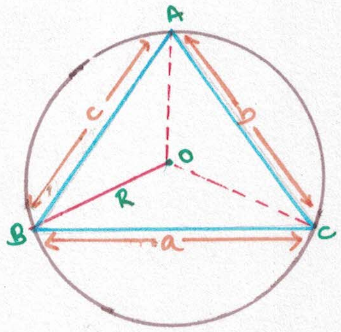
$$\boxed{\Delta = rs}$$

AREA OF TRIANGLE IS:

- $\Delta = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B$
- $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$
- $\Delta = rs$
- $\Delta = \frac{abc}{4R}$
- area $= \frac{a^2}{2} \frac{\sin B \sin C}{\sin A} = \frac{b^2}{2} \frac{\sin A \sin C}{\sin B} = \frac{c^2}{2} \frac{\sin A \sin B}{\sin C}$

DIFFERENT CIRCLES RELATED TO TRIANGLES

1 CIRCUMCIRCLE

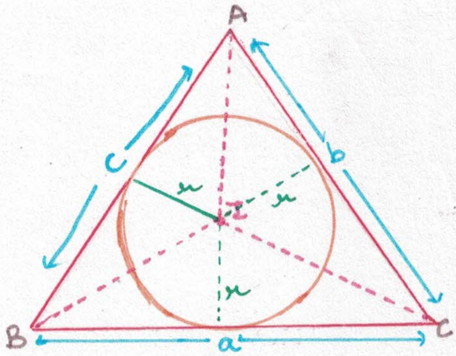


R = circumradius

$$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = R}$$

$$\boxed{R = \frac{abc}{4\Delta}}$$

2 INCIRCLE



r = inradius

area of $\Delta ABC = \Delta$

$= ar(\Delta AIB) + ar(\Delta BIC) + ar(\Delta CIA)$

$\Delta = \frac{1}{2} cr + \frac{1}{2} ar + \frac{1}{2} br$

$\Delta = \frac{1}{2} r(a+b+c)$

$\Delta = rs$

$$\boxed{r = \frac{\Delta}{s}}$$

Inradius = $\frac{\text{area of } \Delta}{\text{Semiperimeter}}$

Replace Δ by heron's formula for area

$$r = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$$

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \times \frac{\sqrt{s-a}}{\sqrt{s-a}}$$

$$r = s-a\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\boxed{r = (s-a)\tan\frac{A}{2}}$$

Similarly, $r = (s-b)\tan\frac{B}{2}$

$r = (s-c)\tan\frac{C}{2}$

$$\boxed{r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}}$$

$$r = \frac{\Delta}{s} = \frac{\Delta}{s}\cdot\frac{\Delta}{\Delta} = \frac{\Delta^2}{s\Delta}$$

$$r = \frac{s(s-a)(s-b)(s-c)}{s\Delta}$$

$$r = \frac{(s-a)(s-b)(s-c)}{abc/4R} \qquad \left[\because \Delta = \frac{abc}{4R}\right]$$

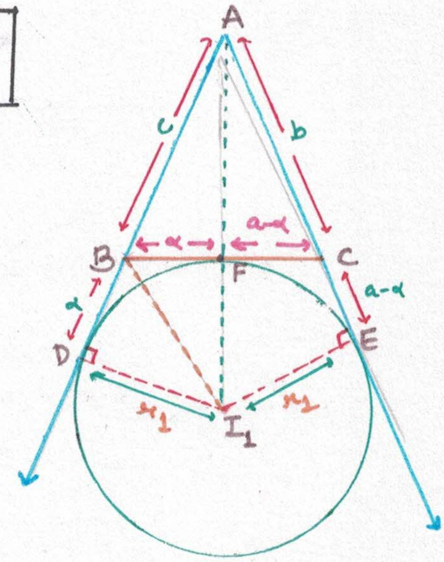
$$r = 4R\sqrt{\frac{(s-a)(s-b)}{ab}} \times \sqrt{\frac{(s-b)(s-c)}{bc}} \times \sqrt{\frac{(s-c)(s-a)}{ac}}$$

$$\boxed{r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

Finally,

$$\boxed{r = \frac{\Delta}{s} = (s-a)\tan\frac{A}{2} = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

3 ESCRIBED CIRCLE / EX-CIRCLE



Area of ADI_1E

$= ar(ADI_1) + ar(AI_1E)$

$= AD \times \frac{r_1}{2} + AE \times \frac{r_1}{2}$ — ①

Area of ΔBFI_1

$= 2 \times ar(BDI_1)$

$= 2 \times \frac{1}{2} \times BD \times r$

$= BDr_1$ — ②

Area of ΔFI_1EC

$= 2 (ar\ CI_1E)$

$= 2 \times \frac{1}{2} \times r_1 \times CE$

$= CEr_1$ — ③

Area of quadri. $BCEI_1$

$=$ ① + ③

$= BDr_1 + CEr_1$

$= r_1 [BD + CE]$

$= r_1 [\alpha + a - \alpha]$

$= r_1 a$ — ④

Area of ΔABC = (area of ADI_1E − area of $BCEI_1$)

$=$ ① − ④

$= \left(\frac{AD + AE}{2}\right) r_1 - r_1 a$

$= \frac{r_1}{2} [AD + AE - 2a]$

$\Delta = \frac{r_1}{2} [c + \alpha + b + a - \alpha - 2a]$

$$\Delta = \frac{r_1}{2}[b+c-a]$$

$$\Delta = r_1 \frac{(2s-2a)}{2}$$

$$\Delta = \frac{2r_1}{2}(s-a)$$

$$\Delta = r_1 (s-a)$$

$$\boxed{r_1 = \frac{\Delta}{s-a}}$$

Similarly, $r_2 = \frac{\Delta}{s-b}$, $r_3 = \frac{\Delta}{s-c}$

$$r_1 = \frac{\Delta}{s-a} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s-a}$$

$$= \sqrt{\frac{s(s-b)(s-c)}{(s-a)} \frac{\times s}{\times s}}$$

$$= s\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$= s \tan\frac{A}{2}$$

$$\boxed{r_1 = s \tan\frac{A}{2}}$$

Similarly, $r_2 = s\tan\frac{B}{2}$, $r_3 = s \tan\frac{C}{2}$

$$r_1 = \frac{\Delta}{s-a} = \frac{\Delta^2}{\Delta(s-a)} = \frac{(s-a)(s-b)(s-c)s}{\Delta(s-a)}$$

$$r_1 = \frac{(s-b)(s-c)s}{abc} \times 4R \qquad \left\{\because \Delta = \frac{abc}{4R}\right\}$$

$$r_1 = \frac{4R(s-b)(s-b)(s-c)}{abc}$$

$$r_1 = 4R\left[\sqrt{\frac{s(s-a)}{bc}} \times \sqrt{\frac{s(s-b)}{ac}} \times \sqrt{\frac{s(s-c)}{ab}}\right]$$

$$\boxed{r_1 = 4R \sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$$

Ques: Prove: $r r_1 r_2 r_3 = \Delta^2$

Sol:
$$(s-a)\tan\frac{A}{2}\cdot\frac{\Delta}{(s-a)}\cdot\frac{\Delta}{(s-b)}\cdot\frac{\Delta}{(s-c)}$$

$$= \frac{\Delta^3}{(s-b)(s-c)}\cdot\tan\frac{A}{2}$$

$$= \frac{\Delta^3}{\sqrt{(s-b)(s-c)}}\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$= \frac{\Delta^3}{\sqrt{s(s-a)(s-b)(s-c)}} = \frac{\Delta^3}{\Delta}$$

$$= \Delta^2$$ Hence Proved.

Ques: Prove: $r_2 r_3 + r_3 r_1 + r_1 r_2 = s^2$

Sol:
$$\left(\frac{\Delta}{s-b}\right)\left(\frac{\Delta}{s-c}\right)+\left(\frac{\Delta}{s-c}\right)\left(\frac{\Delta}{s-a}\right)+\left(\frac{\Delta}{s-a}\right)\left(\frac{\Delta}{s-b}\right)$$

$$= \frac{s(s-a)(s-b)(s-c)}{(s-b)(s-c)}+\frac{s(s-a)(s-b)(s-c)}{(s-\cancel{b}a)(s-c)}+\frac{s(s-a)(s-b)(s-c)}{(s-a)(s-b)}$$

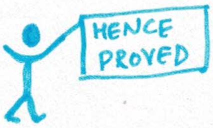
$$= s(s-a)+s(s-b)+s(s-c)$$

$$= s[3s-(a+b+c)]$$

$$= s[3s-2s]$$

$$= s\times s$$

$$= s^2$$



Ques: Prove: $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{r}=0$

Sol: $\frac{s-a}{\Delta}+\frac{s-b}{\Delta}+\frac{s-c}{\Delta}-\frac{s}{\Delta}$

$$= \frac{3s - (a+b+c) - s}{\Delta}$$

$$= \frac{2s - 2s}{\Delta}$$

$$= 0$$

Hence Proved..

Ques: Prove: $\frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{a^2+b^2+c^2}{\Delta^2}$

Sol: $\frac{s^2}{\Delta^2} + \left(\frac{s-a}{\Delta}\right)^2 + \left(\frac{s-b}{\Delta}\right)^2 + \left(\frac{s-c}{\Delta}\right)^2$

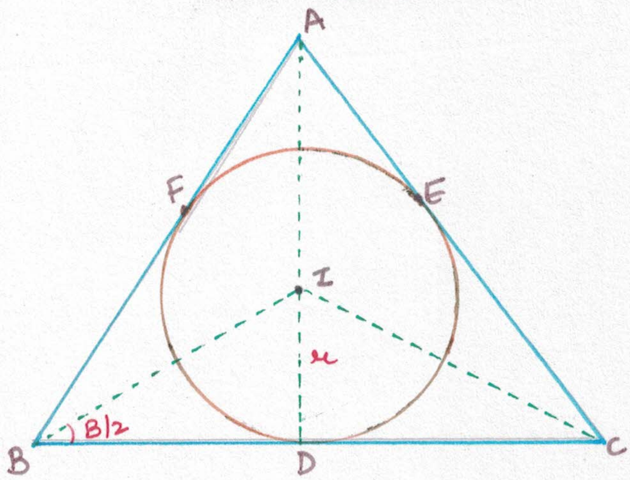
$$= \frac{4s^2 + a^2+b^2+c^2 - 2as - 2bs - 2cs}{\Delta^2}$$

$$= \frac{4s^2 - 2s(2s) + a^2+b^2+c^2}{\Delta^2}$$

$$= \frac{a^2+b^2+c^2}{\Delta^2}$$

Hence Proved

Length of Tangent from Vertex to the Circle



$\frac{ID}{BD} = \tan\frac{B}{2}$

$\frac{r}{BD} = \tan\frac{B}{2}$

$$\boxed{BD = \frac{r}{\tan B/2}}$$

We Know that $\tan\frac{B}{2} \cdot (s-b) = r$

So, $BD = \frac{\tan\frac{B}{2}(s-b)}{\tan B/2}$

$\Rightarrow \boxed{BD = s-b}$ Similarly, $CE = s-c$

$AE = s-a$

So,

$$\boxed{\begin{aligned} BD &= FB = s-b \\ CD &= CE = s-c \\ AE &= AF = s-a \end{aligned}}$$

Length of Angle Bisector

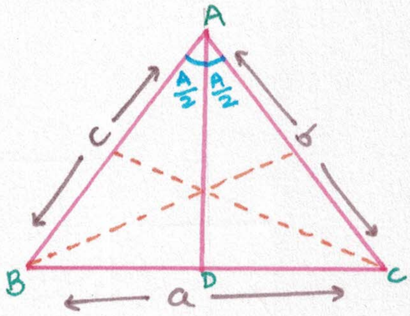
$$\text{area}(\Delta ABC) = \frac{1}{2} bc \sin A \quad — ①$$

$$\text{area}(\Delta ADB) = \frac{1}{2} c \times AD \times \sin\frac{A}{2} \quad — ②$$

$$\text{area}(\Delta ADC) = \frac{1}{2} \times b \times AD \sin\frac{A}{2} \quad — ③$$

Eq. ① = Eq. ② + Eq. ③

$$\frac{1}{2} bc \sin A = \frac{1}{2} c \times AD \times \sin\frac{A}{2} + \frac{1}{2} b \times AD \times \sin\frac{A}{2}$$



$$bc \sin A = AD \sin\frac{A}{2} (b+c)$$

$$AD = \frac{bc \sin A}{\sin A/2 \, (b+c)}$$

$$= \frac{2bc \sin A/2 \cos A/2}{\sin A/2 \, (b+c)}$$

$$\boxed{AD = \frac{2bc \cos A/2}{(b+c)}}$$

Similarly,

$$BE = \frac{2ac \cos B/2}{(a+c)}$$

$$CF = \frac{2ab \cos C/2}{(a+b)}$$

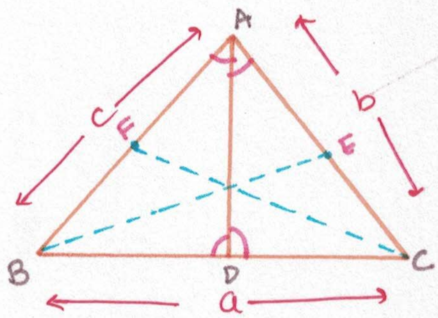
Length of Median

$$\text{area}(ABC) = \frac{1}{2} bc \sin A$$

$$\text{area}(ABD) = \frac{1}{2} c \frac{a}{2} \sin B$$

$$\text{area}(ADC) = \frac{1}{2} b \frac{a}{2} \sin C$$

To find AD,



$$\cos B = \frac{a^2+c^2-b^2}{2ac}$$

$$\cos B = \frac{-AD^2+c^2+(a/2)^2}{2(c)(a/2)}$$

$$\Rightarrow \frac{c^2+\frac{a^2}{4}-AD^2}{ac} = \frac{a^2+c^2-b^2}{2ac}$$

$$\Rightarrow 2c^2+\frac{a^2}{2}-2AD^2 = a^2+c^2-b^2$$

$$\Rightarrow 2c^2+\frac{a^2}{2}-2AD^2 = a^2+c^2-b^2$$

$$\Rightarrow c^2 = a^2-\frac{a^2}{2}-b^2+2AD^2$$

$$\Rightarrow c^2-\frac{a^2}{2}+b^2 = 2AD^2$$

$$\Rightarrow \frac{2c^2+2b^2-a^2}{4} = AD^2$$

$$\Rightarrow \boxed{AD = \frac{1}{2}\sqrt{2c^2+2b^2-a^2}}$$

Similarly,

$$BE = \frac{1}{2}\sqrt{2a^2+2c^2-b^2}$$

$$CF = \frac{1}{2}\sqrt{2b^2+2a^2-c^2}$$

Distance of Circumcentre from sides

1. Circumcentre lies inside the Δ, if Δ is acute angle Δ

② Circumcentre lies on the mid point of hypotenuse if Δ is right angle Δ

③ Circumcentre lies outside the Δ, if Δ is obtuse $\angle$ Δ

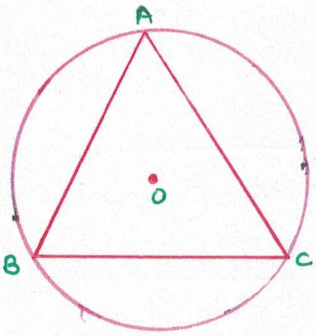
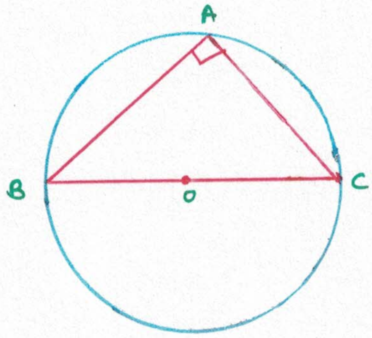
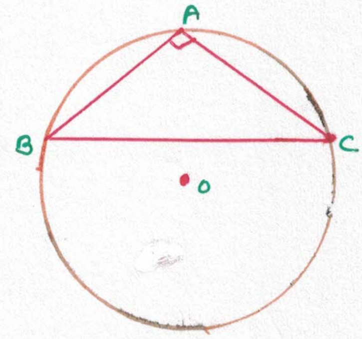


Fig ①



Fig②



Fig③

In ΔBOD,

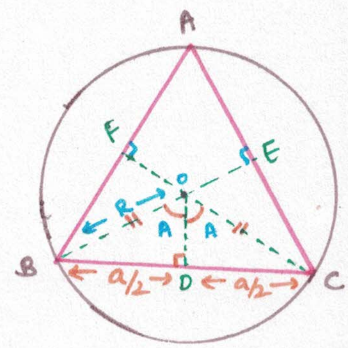
$$\sin A = \frac{a}{2R}$$

$$\cos A = \frac{OD}{R}$$

{ Distance of circumcentre from vertex = circumradius

Similarly,

$$\boxed{\begin{aligned} OD &= |R\cos A| \\ OE &= |R\cos B| \\ OF &= |R\cos C| \end{aligned}}$$



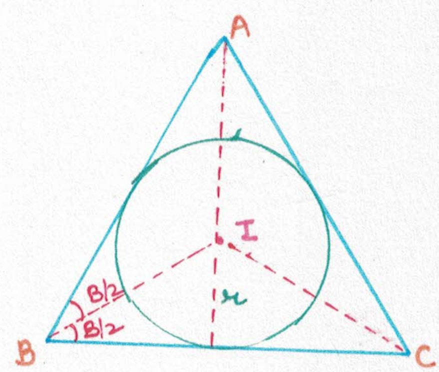
Distance of Incentre from Vertex

Find - AI, BI, CI

$$\sin\frac{B}{2} = \frac{ID}{BI} = \frac{r}{BI}$$

{ Distance of incentre from side = inradius

$$BI = \frac{r}{\sin B/2}$$



Similarly,

$$\boxed{\begin{aligned} CI &= \frac{r}{\sin C/2} \\ AI &= \frac{r}{\sin A/2} \end{aligned}}$$

Regular Polygon

All the sides and angles of a regular polygon are equal

- Equilateral Δ
- Square
- Regular pentagon
- Regular hexagon
- Regular Heptagon

Sum of angles of regular polygon of side 'n' is-

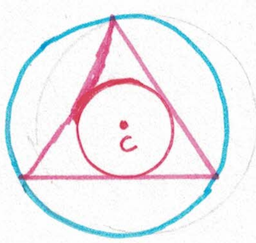
$$\boxed{(n-2)\pi}$$

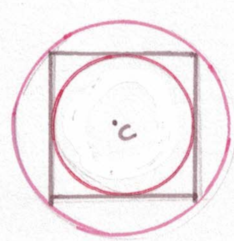
Each angle of regular polygon of side 'n' is -

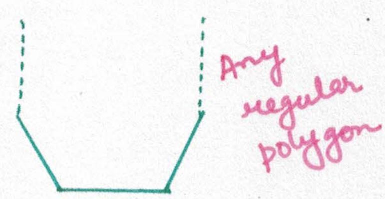
$$\boxed{\frac{(n-2)\pi}{n}}$$

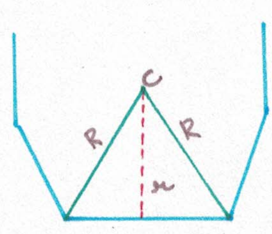
Incenter and Circumcenter of a Regular Polygon

For a regular polygon, incentre and circumcentre will coincide. The point is called centre of the regular polygon.







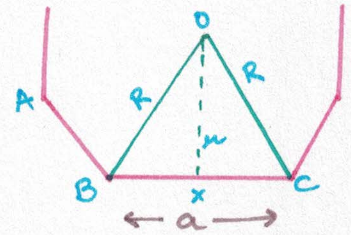


Area of Regular Polygon

n sided regular polygon length of side = a

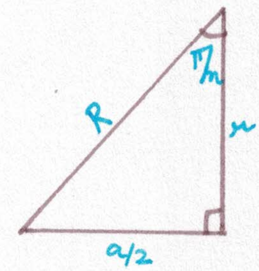
$$\angle BOC = \frac{2\pi}{n}$$

$\angle BOX = \dfrac{\pi}{n}$ $\qquad$ $BX = \dfrac{a}{2}$



$\sin\dfrac{\pi}{n} = \dfrac{a}{2R}$ $\qquad$ $\tan\dfrac{\pi}{n} = \dfrac{a}{2r}$

$$\boxed{\begin{aligned} R &= \frac{a}{2}\operatorname{cosec}\frac{\pi}{n} \\ r &= \frac{a}{2}\cot\frac{\pi}{n} \end{aligned}}$$



Area of $\Delta BOC = \dfrac{1}{2} \times a \times r$

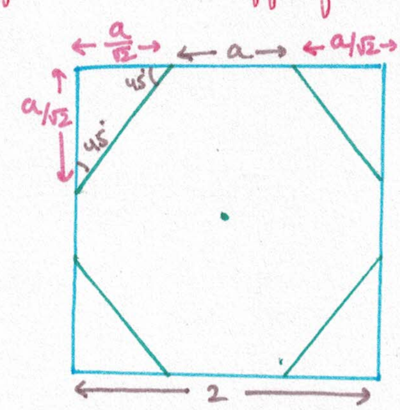
$= \dfrac{1}{2} \times a \times \dfrac{a}{2}\cot\dfrac{\pi}{n}$

Area of such 'n' Δ's $= n \times \dfrac{1}{2} \times \dfrac{a^2}{2}\cot\dfrac{\pi}{n}$

$$\boxed{\text{Area of regular Polygon} = n\left(\frac{a}{2}\right)^2 \cot\frac{\pi}{n}}$$

QUESTIONS

Ques: A square whose side is 2 cm. has its corners cut away so as to form a regular octagon. Find the area of octagon. Also find the dimensions of Δ cut off from the square.



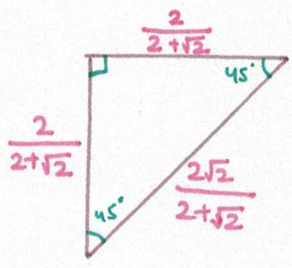
Sol: $\dfrac{a}{\sqrt{2}} + a + \dfrac{a}{\sqrt{2}} = 2$

$a\left(1 + \dfrac{2}{\sqrt{2}}\right) = 2$

$a\left(\dfrac{\sqrt{2}+2}{\sqrt{2}}\right) = 2$

$a = \dfrac{2\sqrt{2}}{\sqrt{2}+2}$ cm.

Dimension of Δ cut off



Area of octgon $= 8 \times \left(\frac{a}{2}\right)^2 \cot\frac{\pi}{8}$

$= 8 \left[\frac{2\sqrt{2}}{(2+\sqrt{2})2}\right]^2 \cot\frac{\pi}{8}$

$= 8\times 2\left[\frac{2-\sqrt{2}}{4-2}\right]^2 \cot\frac{\pi}{8}$

$= \frac{8}{2}(2-\sqrt{2})^2 \cot\frac{\pi}{8}$

$= 4\,(2-1.414)^2 \cot\frac{\pi}{8}$

$= 4 \times 0.546 \cot\frac{\pi}{8}$

$= 2.184\,(\sqrt{3+2\sqrt{2}})$

$= 2.184\,(\sqrt{5.828})$

$\boxed{\text{area} \approx 3.31 \text{ sq. cm.}}$

$\cot\frac{\pi}{8} = ?$

$\cot\frac{\pi}{4} = \frac{1-\tan^2\pi/8}{1+\tan^2\pi/8}$

$\frac{1}{\sqrt{2}} = \frac{1-x}{1+x}$

$\frac{1+\sqrt{2}}{1-\sqrt{2}} = \frac{2x}{-2x}$

$x = \frac{\sqrt{2}-1}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1}$

$x = 2+1-2\sqrt{2}$

$x = 3-2\sqrt{2}$

$\cot^2\frac{\pi}{8} = \frac{1}{3-2\sqrt{2}} \times \frac{3+2\sqrt{2}}{3+2\sqrt{2}}$

$= \frac{3+2\sqrt{2}}{9-8}$

$= 3+2\sqrt{2}$

Ques: Of two regular polygons of 'n' sides, one circumscribed and the other is inscribed in a given circle. Prove that the perimeter of circumscribed polygon, the circle and the inscribed polygon are in the ratio -

$\sec\frac{\pi}{n} : \frac{\pi}{n}\operatorname{cosec}\frac{\pi}{n} : 1$

Sol: Side of smaller polygon $\rightarrow a_1$

Radius of circumscribed circle $\rightarrow \frac{a_1}{2}\operatorname{cosec}\frac{\pi}{n}$

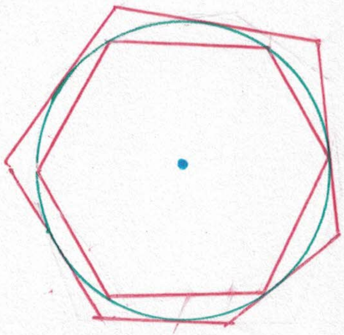
Side of bigger polygon $\rightarrow a_2$

Radius of inscribed circle $\rightarrow \frac{a_2}{2}\cot\frac{\pi}{n}$

$\because$ The circle is same

$$\frac{a_1}{2}\operatorname{cosec}\frac{\pi}{n} = \frac{a_2}{2}\cot\frac{\pi}{n}$$

$$\frac{a_1}{\sin \pi/n} = a_2 \frac{\cos \pi/n}{\sin \pi/n}$$



$$a_1 = a_2 \cos \pi/4$$

$$a_1 \sec \frac{\pi}{n} = a_2$$

Perimeter of bigger polygon (P_1) = na_2

$= na_1 \sec \pi/n$

Perimeter of circle (P_2) = $2\pi r$

$= 2\pi \frac{a_1}{2} \operatorname{cosec} \frac{\pi}{n}$

Perimeter of smaller polygon (P_3) = na_1

$$P_1 : P_2 : P_3 = na_1 \sec \frac{\pi}{n} : \pi a_1 \operatorname{cosec} \frac{\pi}{n} : na_1$$

$$\boxed{P_1 : P_2 : P_3 = \sec \frac{\pi}{n} : \frac{\pi}{n} \operatorname{cosec} \frac{\pi}{n} : 1}$$

Ques: Two regular polygon of n and $2n$ sides have the same perimeter, Show that their areas are in the ratio-

$$2 \cos \frac{\pi}{n} : 1 + \cos \frac{\pi}{n}$$

Sol: Perimeter $\rightarrow x$

$$\text{Area}_1 = n\left(\frac{x}{2n}\right)^2 \cot \frac{\pi}{n} \qquad \text{Area}_2 = 2n\left(\frac{x}{2 \times 2n}\right)^2 \cot \frac{\pi}{2n}$$

$$= \frac{x^2}{4n} \cot \frac{\pi}{n} \qquad = \frac{x^2}{8n} \cot \frac{\pi}{2n}$$

$A_1 : A_2$

$$\Rightarrow \frac{x^2}{4n} \cot \frac{\pi}{n} : \frac{x^2}{8n} \cot \frac{\pi}{2n}$$

$$\Rightarrow \frac{2 \cos \pi/n}{\sin \pi/n} : \frac{\cos \pi/2n}{\sin \pi/2n}$$

$$\Rightarrow \quad 2\cos\frac{\pi}{n} \; : \; \frac{\cos\frac{\pi}{2n}\,\sin\frac{\pi}{n}}{\sin\frac{\pi}{2n}}$$

$$\Rightarrow \quad 2\cos\frac{\pi}{n} \; : \; \frac{\cos\frac{\pi}{2n}\cdot 2\sin\frac{\pi}{2n}\,\cos\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}$$

$$\Rightarrow \quad 2\cos\frac{\pi}{n} \; : \; 2\cos^2\frac{\pi}{2n}$$

$$\Rightarrow \quad \boxed{2\cos\frac{\pi}{n} \; : \; 1+\cos\frac{\pi}{n}}$$

Ques: side $\rightarrow 2a$, 'n' sided polygon, $R+r = ?$

Sol: $R = \frac{2a}{2}\operatorname{cosec}\frac{\pi}{n} = a\operatorname{cosec}\frac{\pi}{n}$

$r = \frac{2a}{2}\cot\frac{\pi}{n} = a\cot\frac{\pi}{n}$

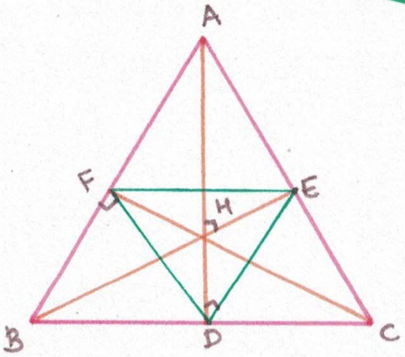
$$R+r = a\left(\operatorname{cosec}\frac{\pi}{n} + \cot\frac{\pi}{n}\right)$$

$$= a\left(\frac{1+\cos\frac{\pi}{n}}{\sin \pi/n}\right)$$

$$= a\left(\frac{2\cos^2\frac{\pi}{2n}}{2\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}}\right)$$

$$\boxed{R+r = a\cot\frac{\pi}{2n}}$$

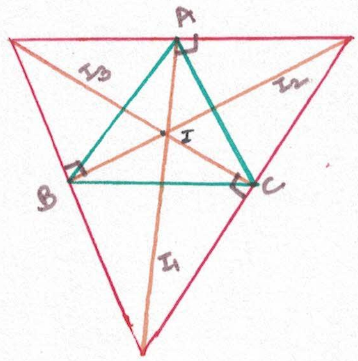
ORTHOCENTER & PEDAL TRIANGLE



D, E and F are foots of $\perp$ from the vertices on opposite sides.

The ΔDEF is called Pedal triangle of ΔABC.

EXCENTRAL TRIANGLE



The triangle whose vertices are the ex- centres of ΔABC is called excentral triangle of ΔABC

$\Delta I_1 I_2 I_3$ is excentral Δ

$\angle ICI_1 = \frac{\pi}{2} \Rightarrow I_3C \perp I_1I_2$

Similarly, $I_2B \perp I_1I_3$

and, $I_1A \perp I_2I_3$

A, B, C are foot of perpendiculars.

So, we can say,

ΔABC is pedal Δ to $\Delta I_1I_2I_3$

Angles of ex-central Δ

$\angle I_1$, $\angle I_2$, $\angle I_3$

$\angle CBI_1 = 90 - \frac{B}{2}$ $\qquad$ $\angle BCI_1 = 90 - \frac{C}{2}$

in ΔBCI_1

$\angle B + \angle C + \angle I_1 = 180^\circ$

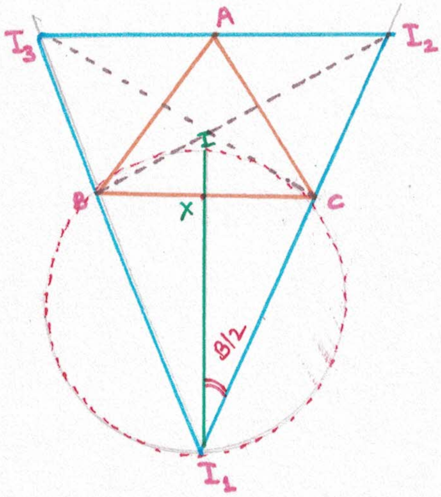
$\angle I_1 = 180^\circ - 90 + \frac{B}{2} - 90 + \frac{C}{2}$

$\angle I_1 = \frac{\angle B + \angle C}{2} = \frac{\pi}{2} - \frac{A}{2}$

Similarly,

$$\boxed{\begin{aligned} \angle I_2 &= \frac{\angle A + \angle C}{2} = \frac{\pi}{2} - \frac{B}{2} \\ \angle I_3 &= \frac{\angle A + \angle B}{2} = \frac{\pi}{2} - \frac{C}{2} \end{aligned}}$$

Distance between Incentre and Excentre



For II_1

$\angle I_1 = \frac{\pi}{2} - \frac{\angle A}{2}$

In ΔIXC

$IC = \frac{r}{\sin\left(\frac{C}{2}\right)}$

$\square BICI_1$ is cyclic

In ΔII_1C,

$\frac{IC}{II_1} = \sin\left(\frac{B}{2}\right)$ $[\because \angle II_1C = B/2]$

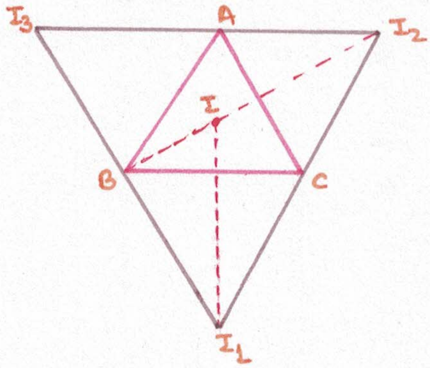
$$II_1 = \frac{IC}{\sin B/2} = \frac{r}{\sin B/2 \ \sin C/2}$$

$$II_1 = \frac{4R \sin A/2 \ \sin B/2 \ \sin C/2}{\sin B/2 \ \sin C/2}$$

Similarly,

$$\boxed{\begin{aligned} II_1 &= 4R \sin A/2 \\ II_2 &= 4R \sin B/2 \\ II_3 &= 4R \sin C/2 \end{aligned}}$$

Sides of Excentral Triangle



In $\Delta I_1 I I_2$

$\angle II_1I_2 = \frac{\angle B}{2}$

$\angle I_1I_2I = \frac{\angle A}{2}$

$\Rightarrow \angle I_1 I I_2 = \pi - \left(\frac{A+B}{2}\right)$

Apply sine rule,

$$\frac{II_1}{I_1I_2} = \frac{\sin I_2}{\sin I} = \frac{\sin A/2}{\sin\left(\pi - \left(\frac{A+B}{2}\right)\right)}$$

$$\frac{4R \sin \frac{A}{2}}{I_1 I_2} = \frac{\sin \frac{A}{2}}{\cos C/2}$$

$$I_1 I_2 = 4R \cos\frac{C}{2}$$
$$I_1 I_3 = 4R \cos\frac{B}{2}$$
$$I_2 I_3 = 4R \cos\frac{A}{2}$$

PRACTISE PROBLEM

1. Prove that:

$$IA \cdot IB \cdot IC = abc \tan\frac{A}{2} \tan\frac{B}{2} \tan\frac{C}{2}$$

Sol:

$$\frac{r}{\sin A/2} \cdot \frac{r}{\sin B/2} \cdot \frac{r}{\sin C/2}$$

$$= r^2 \frac{4R \sin A/2 \sin B/2 \sin C/2}{\sin A/2 \sin B/2 \sin C/2}$$

$$= r^2 4R$$

$$= \frac{abc}{\Delta} \left(\frac{\Delta}{s}\right)^2 = \frac{\Delta\, abc}{s^2}$$

$$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s^2} \cdot abc$$

$$= abc \sqrt{\frac{(s-a)(s-b)(s-c)\,(s-a)(s-b)(s-c)\, s}{(s-a)\,(s-b)\,(s-c)\, s^4}}$$

$$= abc \tan\frac{A}{2} \cdot \tan\frac{B}{2} \cdot \tan\frac{C}{2}$$

2. Prove that:

$$II_1 \cdot II_2 \cdot II_3 = 16R^2 r$$

Sol:

$$4R \sin\frac{A}{2} \cdot 4R \sin\frac{B}{2} \cdot 4R \sin\frac{C}{2}$$

$$= (4R)^3 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$$

$$= 16R^2 r$$

3. Prove that:

$$II_1^2 + I_2 I_3^2 = II_2^2 + I_3 I_1^2 = II_3^2 + I_1 I_2^2$$

Sol: I) $\left(4R\sin\frac{A}{2}\right)^2 + \left(4R\cos\frac{A}{2}\right)^2$

$= (4R)^2\left[\sin^2\frac{A}{2} + \cos^2\frac{A}{2}\right]$

$= (4R)^2$

II) $\left(4R\sin\frac{B}{2}\right)^2 + \left(4R\cos\frac{B}{2}\right)^2$

$= (4R)^2\left[\sin^2\frac{B}{2} + \cos^2\frac{B}{2}\right]$

$= (4R)^2$

III) $\left(4R\sin\frac{C}{2}\right)^2 + \left(4R\cos\frac{C}{2}\right)^2$

$= (4R)^2\left[\sin^2\frac{C}{2} + \cos^2\frac{C}{2}\right]$

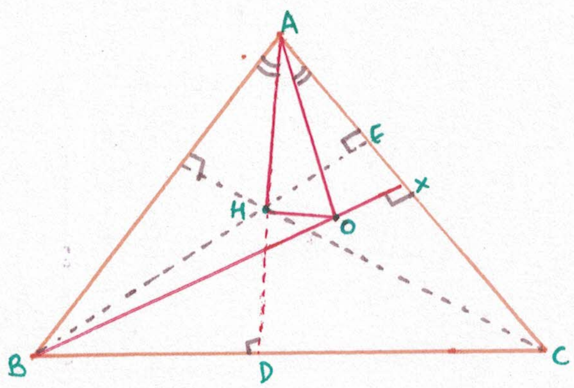
$= (4R)^2$

$\boxed{I = II = III}$ Hence Proved.

DISTANCE BETWEEN DIFFERENT CENTRES

1 Distance between circumcentre and Orthocentre:

$R = \sqrt{1 - 8\cos A \cos B \cos C}$



$\angle BAD = 90 - B$

O is circumcentre

$\angle AOC = 2\angle B$

$OX \perp AC$,

So, $\angle AOX = \frac{\angle AOC}{2} = \frac{2\angle B}{2} = \angle B$

In ΔAOE,

$\angle OAE = 90 - B$

$\angle HAO = \angle A - (90 - B + 90 - B)$

$= A - 180 + 2B$

$= A - A - B - C + 2B$

$\angle HAO = B - C$

Length $HO = R$

In ΔAHO,

$$\cos(HAO) = \frac{AH^2 + AO^2 - OH^2}{2AH \cdot AO}$$

$$\cos(B-C) = \frac{(2R\cos A)^2 + R^2 - OH^2}{2(2R\cos A)(R)}$$

$\cos(B-C) \cdot 4R^2\cos A = 4R^2\cos^2 A + R^2 - OH^2$

$OH^2 = 4R^2\cos^2 A + R^2 - 4R^2\cos A\cos(B-C)$

$OH^2 = R^2 + 4R^2\cos A(\cos A - \cos(B-C))$

$OH^2 = R^2 + 4R^2\cos A(-\cos(B+C) - \cos(b-c))$

$OH^2 = R^2 - 4R^2\cos A(\cos(B+C) + \cos(B-C))$

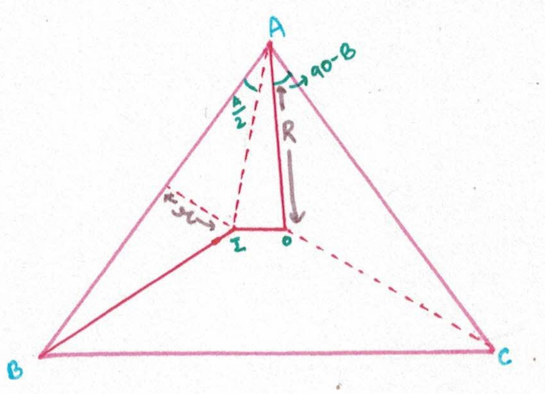
$OH^2 = R^2 - 4R^2\cos A(2\cos B\cos C)$

$OH^2 = R^2 - 8R^2\cos A\cos B\cos C$

$OH = R\sqrt{1 - 8\cos A\cos B\cos C}$

2 **Distance between circumcentre and incentre:**

$\sqrt{R^2 - 2Rr}$



$\angle IAO = A - \left(\frac{A}{2} + 90 - B\right)$

$= \frac{A}{2} + B - 90$

$= \frac{A}{B} + B - \frac{A}{2} - \frac{B}{2} - \frac{C}{2}$

$\angle IAO = \frac{B}{2} - \frac{C}{2}$

$IA = \frac{r}{\sin A/2}$ $\quad OA = R$

In ΔIAO,

$$\cos \angle IAO = \frac{IA^2 + OA^2 - IO^2}{2IA \cdot OA}$$

$$\cos\left(\frac{B-C}{2}\right) = \frac{\frac{r^2}{\sin^2 A/2} + R^2 - IO^2}{2\left(\frac{r}{\sin A/2}\right) \cdot R}$$

$$IO^2 = \frac{r^2}{\sin^2 \frac{A}{2}} + R^2 - \cos\left(\frac{B-C}{2}\right) \cdot \frac{2rR}{\sin A/2}$$

Write $r = 4R \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$

$$IO^2 = (4R)^2 \sin^2\frac{B}{2} \sin^2\frac{C}{2} + R^2 - \cos\left(\frac{B-C}{2}\right) \cdot 2R \sin\frac{B}{2} \sin\frac{C}{2}$$

$$IO^2 = R^2 + 8R^2 \left[\sin\frac{B}{2} \sin\frac{C}{2}\right]\left[2\sin\frac{B}{2}\sin\frac{C}{2}\right] - \cos\left(\frac{B-C}{2}\right)$$

$$IO^2 = R^2 + 8R^2 \sin\frac{B}{2}\sin\frac{C}{2}\left[-\cos\left(\frac{B+C}{2}\right) + \cos\left(\frac{B-C}{2}\right) - \cos\left(\frac{B-C}{2}\right)\right]$$

$$IO^2 = R^2 + 8R^2 \sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{A}{2}$$

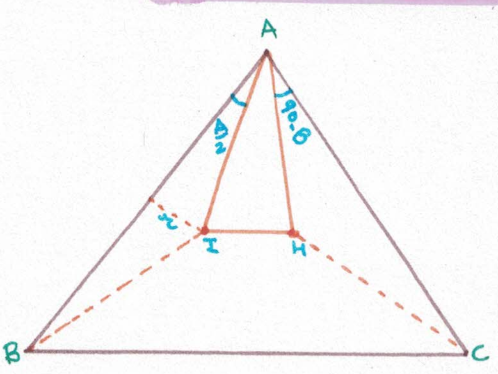
$$IO^2 = R^2 - 2R\left[4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right]$$

$$IO^2 = R^2 - 2Rr$$

$$IO = \sqrt{R^2 - 2Rr}$$

3 **Distance between incentre and orthocenter:**

$$\sqrt{2r^2 - 4R^2 \cos A \cos B \cos C}$$



$$AI = \frac{r}{\sin\frac{A}{2}}$$

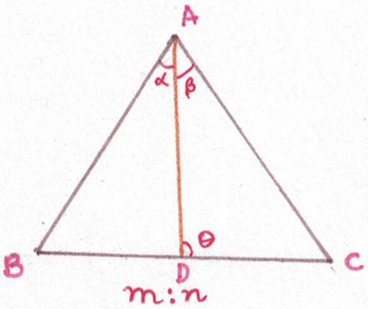
$$\angle A = \frac{B-C}{2} \qquad AH = 2R\cos A$$

$$\cos\left(\frac{B-C}{2}\right) = \frac{\frac{r^2}{\sin^2 A/2} + (2R\cos A) + IH^2}{2r/\sin A/2 \cdot 2R\cos A}$$

$$IH^2 = (4R)^2 \sin^2\frac{B}{2} \cdot \sin^2\frac{C}{2} + (4R)^2 \cos^2 A - (4R)^2 \sin\frac{A}{2} \sin\frac{C}{2} \cos\left(\frac{B-C}{2}\right)$$

$$IH^2 = 4R^2\left[\cos^2 A + 4R \sin\frac{B}{C} \sin\frac{C}{2}\left(\sin\frac{B}{2}\sin\frac{C}{2} - \cos\left(\frac{B-C}{2}\right)\right)\right.$$

m-n cot THEOREM



$BD : DC = m : n$

$\angle BAD = \alpha$ $\quad \angle DAC = \beta$

and $\angle ADC = \theta$ then,

① $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

In ΔABD,

$\angle B = \pi - (\pi - \theta + \alpha)$

$\angle B = \theta - \alpha$

$$\frac{BD}{AD} = \frac{\sin\alpha}{\sin(\theta-\alpha)} \quad — ①$$

In ΔADC,

$\angle C = \pi - \theta - \beta$

$\angle C = \pi - (\theta + \beta)$

$$\frac{CD}{AD} = \frac{\sin\beta}{\sin(\theta+\beta)} \quad — ②$$

$$\frac{BD}{CD} = \frac{\sin\alpha \ \sin(\theta+\beta)}{\sin\beta \cdot \sin(\theta-\alpha)}$$

$$\frac{BD}{CD} = \frac{\sin\alpha(\sin\theta\cos\beta + \cos\theta\sin\beta)}{\sin\beta(\sin\theta\cos\alpha - \cos\theta\sin\alpha)}$$

$$\frac{m}{n} = \frac{\tan\alpha\tan\theta + \tan\alpha\tan\beta}{\tan\beta\tan\theta - \tan\beta\tan\alpha}$$

$$\frac{m}{n} = \frac{\cot\beta + \cot\theta}{\cot\alpha - \cot\theta}$$

$m(\cot\alpha - \cot\theta) = n(\cot\beta + \cot\theta)$

$$m\cot\alpha - n\cot\beta = (m+n)\cot\theta$$

$$\boxed{(m+n)\cot\theta = m\cot\alpha - n\cot\beta}$$

② $(m+n)\cot\theta = n\cot B - m\cot C$

$$\angle BAD = 180 - B - \angle BDA$$

$$= 180 - B - (180-\theta)$$

$$\angle BAD = \theta - B$$

$$\angle DAC = 180 - (\theta + C)$$

In ΔABD,

$$\frac{BD}{\sin\angle BAD} = \frac{AD}{\sin B}$$

$$\frac{BD}{AD} = \frac{\sin(\theta - B)}{\sin B} \quad \text{—— ③}$$

In ΔADC,

$$\frac{DC}{\sin\angle DAC} = \frac{AD}{\sin C}$$

$$\frac{DC}{AD} = \frac{\sin[180-(\theta+C)]}{\sin C} \quad \text{—— ④}$$

Divide ① by ②,

$$\frac{BD}{DC} = \frac{\sin(\theta-B)\cdot\sin C}{\sin B\cdot\sin(\theta+C)}$$

$$\frac{m}{n} = \frac{[\sin\theta\cos B - \cos\theta\sin B]\sin C}{\sin B(\sin\theta\cos C - \cos\theta\sin C)}$$

Multiply & Divide by $\sin\theta \sin C$

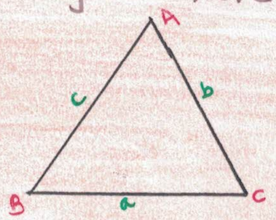
$$\frac{m}{n} = \frac{\cot B - \cot\theta}{\cot C - \cot\theta}$$

$$m\cot C - m\cot\theta = n\cot B - n\cot\theta$$

$$\boxed{(m+n)\cot C = n\cot B - m\cot C}$$

SOLUTION OF TRIANGLE

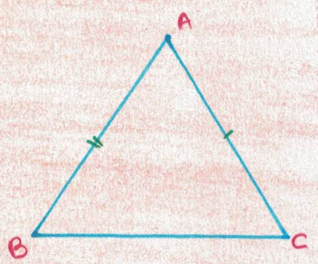
1) If three sides (a, b, c) are given, we need to find A, B, C use cosine rule,



$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

2) If 2 sides and included angle are given. b, C and $\angle A$ are given,

a) use cosine rule
b) Use napier's analogy



$$\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2}$$

$\Downarrow$

We'll get B−C

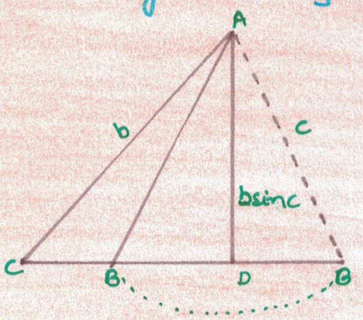
we already Know $B+C = \pi - A$

So, we get B, C

3) Ambiguous case

If 2 sides and one angle [which is not made by 2 sides] are given,

b, c, $\angle C$



a) When $c < b \sin A$, no Δ possible

b) When $c = b \sin C$, one Δ possible

$\angle B = 90^\circ$

c) When $b \sin A < c < b$ two Δ possible

d) When $c > b$, one possible Δ